

# ON A COALGEBRAIC VIEW ON LOGIC

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**ABSTRACT.** In this paper we present methods of transition from one perspective on logic to others, and apply this in particular to obtain a coalgebraic presentation of logic. The central ingredient in this process is to view consequence relations as morphisms in a category.

## INTRODUCTION

Logic is a very important concept in several areas of mathematics to rigorously accommodate the principles of sound reasoning. We can consider two main subareas: (1) the theoretical, such as foundations, model theory, philosophy and formal linguistics, and (2) the applied such as computer science where several logical system have been used to develop formal methods for verification and specification of software systems (see [11]). For a nice discussion of notions of general abstract logic we refer to [4] (see in particular to [3]). In this paper we wish to participate in this discussion by presenting universal logic as a covariety of coalgebras for an appropriate functor. This is achieved by viewing consequence relations as morphisms in a category.

This work continues the line of research started by A. Palmigiano in [12] that studies the connection of the theory of consequence operators (from a logic perspective as in [5, 15]) and the theory of special abstract mathematical structures, namely coalgebras and dialgebras. Palmigiano presented the translation of some basic notions of the theory of consequence operators into notions of the theory of coalgebras, which is closely related to the well known way topological spaces can be considered as coalgebra for the filter functor discussed in [6]. In this paper we present a slightly different construction which is closer to the one for topological spaces (see Remark 14).

**Outline of the paper.** We present logic (to be more precise: a consequence relation) on an abstract set rather than a set of formulæ. We exhibit various ways to encode this structure which have roots in different fields of mathematics, namely topology and (co)algebra. We start by taking a relational approach to the consequence relation; we define appropriated identity and composition to form a monoid that represents the abstract

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logic. We discuss properties of these structures and of maps between them that can be expressed by simply (in)equalities with the help of suitable defined compositions, which is useful when transporting notions or ideas from one structure to the other since the transition maps preserve both composition and inequalities.

A similar work is carried on with other two perspectives: topological and coalgebraic. In the former, based on the well known connection between consequence relations and closure operators, we discuss the meaning of special maps between closure spaces in the context of abstract logic. Relating to the latter, motivated by the fact that topological spaces can be seen as coalgebras for the filter functor [6] and by the work of Palmigiano refereed above, we show how an abstract algebraic logic can be seen as a coalgebra for a natural functor. We develop a detailed analysis of the maps between coalgebras and their meaning concerning the underlying abstract logic. We close the paper by presenting an elegant and simple proof that the class of coalgebras coming from abstract logics forms a covariety.

Our presentation here rests partially on general results of [13].

## 1. A RELATIONAL VIEW ON LOGIC.

A *consequence relation*  $\vdash$  on a set  $X$  is a relation  $\vdash: PX \rightarrow X$  between subsets of  $X$  and points of  $X$  which satisfies

- (1) if  $x \in A$ , then  $A \vdash x$ ,
- (2) if  $A \vdash x$  and  $A \subseteq B$ , then  $B \vdash x$ , and
- (3) if  $A \vdash y$  for all  $y \in B$  and  $B \vdash x$ , then  $A \vdash x$ ;

for all  $A, B \subseteq X$  and  $x \in X$ . In other words, one requires the reflexivity, weakening and cut rule but cannot anymore insist on structurality simply because our “formulae” are now structureless points of an abstract set. Thanks to the second condition above, one can substitute the first one by

- (1')  $\{x\} \vdash x$  for all  $x \in X$ .

The pair  $(X, \vdash)$  one calls an *abstract logic*. Given also a set  $Y$  with a consequence relation  $\Vdash$  and a map  $f: X \rightarrow Y$ , one says that  $f$  is *consequence preserving* whenever  $A \vdash x$  implies  $f(A) \Vdash f(x)$ , for all  $A \subseteq X$  and  $x \in X$ ; and  $f$  is called *conservative* if  $A \vdash x \iff f(A) \Vdash f(x)$ .

The axioms defining a consequence relation can be elegantly expressed using the calculus of relations as we explain next. Recall that for relations  $r: X \rightarrow Y$  and  $s: Y \rightarrow Z$ , one calculates the composite relation  $s \cdot r: X \rightarrow Z$  as

$$x(s \cdot r)z \iff \exists y \in Y (x r y) \& (y s z).$$

Every function can be seen as a relation, and relational composition is actually function composition if  $s$  and  $r$  are functions. Since the identity function  $1_X: X \rightarrow X$  acts as an identity with respect to relational composition, one obtains the category  $\mathbf{Rel}$  of sets and relations. It is worth noting that  $\mathbf{Rel}$  is actually an *ordered category* since inclusion defines an order relation on the set  $\mathbf{Rel}(X, Y)$  of relations from  $X$  to  $Y$ , and composition from

either side preserves this order. One has a functor

$$\mathbf{Set} \rightarrow \mathbf{Rel}$$

which interprets every function as a relation, and also a functor

$$\mathbf{Set}^{\text{op}} \rightarrow \mathbf{Rel}$$

which takes every function  $f : X \rightarrow Y$  to its “inverse image relation”  $f^\circ : Y \multimap X$  defined by  $y f^\circ x$  whenever  $f(x) = y$ . It is worth noting that  $f$  and  $f^\circ$  form an adjunction  $f \dashv f^\circ$  in  $\mathbf{Rel}$ , meaning that one has the inequalities

$$1_X \subseteq f^\circ \cdot f \quad \text{and} \quad f \cdot f^\circ \subseteq 1_Y.$$

This fact is in particular useful since it allows to “shuffle functions around” in inequalities involving their composition with relations.

**Lemma 1.** *Let  $f : X \rightarrow Y$  be a function and  $r : A \multimap X$ ,  $s : A \multimap Y$ ,  $r' : X \multimap B$  and  $s' : Y \multimap B$  be relations. Then*

$$f \cdot r \subseteq s \iff r \subseteq f^\circ \cdot s \quad \text{and} \quad r' \subseteq s' \cdot f \iff r' \cdot f^\circ \subseteq s'.$$

*Proof.* If  $f \cdot r \subseteq s$ , then  $r \subseteq f^\circ \cdot f \cdot r \subseteq f^\circ \cdot s$ ; and from  $r \subseteq f^\circ \cdot s$  one gets  $f \cdot r \subseteq f \cdot f^\circ \cdot s \subseteq s$ . The second equivalence one obtains similarly.  $\square$

Every relation  $r : X \multimap Y$  can be lifted to a relation  $\hat{P}r : PX \multimap PY$  between the powersets of  $X$  and  $Y$  via

$$A(\hat{P}r)B \text{ whenever } \forall y \in B \exists x \in A. x r y,$$

for all  $A \subseteq X$  and  $B \subseteq Y$ . In the sequel we will write  $P : \mathbf{Set} \rightarrow \mathbf{Set}$  for the powerset functor which sends each set  $X$  to its power set  $PX$ , and a function  $f : X \rightarrow Y$  to the “direct image function”  $Pf : PX \rightarrow PY$  where  $Pf(A) = \{f(x) \mid x \in A\}$ , for every  $A \subseteq X$ . We note that  $P$  is actually part of a monad (see [7] for details)  $(P, e, m)$  where  $e_X : X \rightarrow PX$ ,  $x \mapsto \{x\}$  and  $m_X : PPX \rightarrow PX$ ,  $\mathcal{A} \mapsto \bigcup \mathcal{A}$ . This means in particular that  $e = (e_X)$  and  $m = (m_X)$  are natural transformations, so that for every function  $f : X \rightarrow Y$  one has  $e_Y \cdot f = Pf \cdot e_X$  and  $Pf \cdot m_X = m_Y \cdot PPf$ . Using Lemma 1, from the latter one obtains  $PPf \cdot m_X^\circ \subseteq m_Y^\circ \cdot Pf$ , and it is not hard to see that one even has equality  $PPf \cdot m_X^\circ = m_Y^\circ \cdot Pf$  again.

**Proposition 2** ([13, 14]). *The following assertions hold:*

- (1) *For all relations  $r : X \multimap Y$  and  $s : Y \multimap Z$ ,  $\hat{P}(s \cdot r) = (\hat{P}s) \cdot (\hat{P}r)$ .*
- (2) *For every function  $f : X \rightarrow Y$ ,  $Pf \subseteq \hat{P}f$  and  $(Pf)^\circ \subseteq \hat{P}(f^\circ)$ .*
- (3) *For every function  $f : X \rightarrow Y$  and relations  $s : Y \multimap B$  and  $r : A \multimap Y$ ,  $\hat{P}(s \cdot f) = \hat{P}s \cdot Pf$  and  $\hat{P}(f^\circ \cdot r) = (Pf)^\circ \cdot \hat{P}r$ .*
- (4) *For every relation  $r : X \multimap Y$ ,  $e_Y \cdot r \subseteq \hat{P}r \cdot e_X$  and  $\hat{P}r \cdot m_X = m_Y \cdot \hat{P}\hat{P}r$ .*

We can rewrite now the axioms of a consequence relation as simple *reflexivity* and *transitivity* conditions:

$$\{x\} \vdash x \quad \text{and} \quad (\mathcal{A}(\hat{P}\vdash)A \ \& \ A a x) \Rightarrow (\bigcup \mathcal{A}) a x,$$

for all  $\mathcal{A} \in PPX$ ,  $A \in PX$  and  $x \in X$ . Equivalently, and without referring to points, these conditions read as

$$1_X \subseteq (\vdash \cdot e_X) \quad \text{and} \quad (\vdash \cdot \hat{P}\vdash) \subseteq (\vdash \cdot m_X).$$

Furthermore, using Lemma 1, these conditions become

$$e_X^\circ \subseteq \vdash \quad \text{and} \quad (\vdash \cdot (\hat{P}\vdash) \cdot m_X^\circ) \subseteq \vdash.$$

In general, for relations  $r : PX \multimap Y$  and  $s : PY \multimap Z$ , we can think of

$$(*) \quad s \circ r := s \cdot (\hat{P}r) \cdot m_X^\circ$$

as a kind of composite relation  $s \circ r : PX \multimap Z$ . This composition is associative and has the relations  $\Delta_X : PX \multimap X$  defined by  $A \Delta_X x \iff x \in A$  as “weak” identities since

$$r \subseteq \Delta_Y \circ r \quad \text{and} \quad r \subseteq r \circ \Delta_X.$$

One has even equality above if and only if  $r : PX \multimap Y$  is *monotone*, that is,  $Ary$  and  $A \subseteq B$  imply  $Br y$ . For every set  $X$ ,  $\Delta_X : PX \multimap X$  is monotone, and the composite  $s \circ r$  of monotone relations is again monotone. Therefore we can form the category **MRel** having sets as objects, a morphism  $r : X \multimap Y$  in **MRel** is a monotone relation  $r : PX \multimap Y$  whose composite  $s \circ r : X \multimap Z$  with  $s : Y \multimap Z$  is defined by (\*), and  $\Delta_X : X \multimap X$  is the identity morphism on  $X$ . All told:

**Proposition 3.** *A relation  $\vdash : PX \multimap X$  is a consequence relation on  $X$  if and only if  $\vdash : X \multimap X$  is a monoid in **MRel** with unit  $\Delta_X \subseteq \vdash$  and multiplication  $(\vdash \circ \vdash) \subseteq \vdash$ .*

To every function  $f : X \rightarrow Y$  we associate monotone relations

$$f_\# := e_Y^\circ \cdot \hat{P}f : PX \multimap Y \quad \text{and} \quad f^\# := e_X^\circ \cdot \hat{P}(f^\circ) : PY \multimap X.$$

Note that  $(1_X)_\# = \Delta_X = 1_X^\#$  and, more generally,  $f_\# = \Delta_Y \cdot Pf$  and  $f^\# = f^\circ \cdot \Delta_Y$ , hence  $A f_\# y \iff y \in f(A)$  and  $B f^\# x \iff x \in f^{-1}(B)$ , for all  $A \subseteq X$ ,  $B \subseteq Y$ ,  $x \in X$  and  $y \in Y$ .

**Lemma 4.** *Let  $f : X \rightarrow Y$  be a function and  $r : PZ \multimap Y$  and  $s : PY \multimap Z$  be monotone relations. Then*

$$f^\# \circ r = f^\circ \cdot r \quad \text{and} \quad s \circ f_\# = s \cdot Pf,$$

*Proof.* We calculate

$$f^\# \circ r = f^\circ \cdot \Delta_Y \cdot \hat{P}r \cdot m_Z^\circ = f^\circ \cdot (\Delta_Y \circ r) = f^\circ \cdot r$$

and

$$\begin{aligned} s \circ f_\# &= s \cdot \hat{P}(\Delta_Y \cdot Pf) \cdot m_X^\circ = s \cdot \hat{P}\Delta_Y \cdot PPf \cdot m_X^\circ \\ &= s \cdot \hat{P}\Delta_Y \cdot m_Y^\circ \cdot Pf = (s \circ \Delta_Y) \cdot Pf = s \cdot Pf. \quad \square \end{aligned}$$

**Corollary 5.** *Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be functions. Then the following assertions hold.*

- (1)  $(g \cdot f)_\# = g_\# \circ f_\#$  and  $(g \cdot f)^\# = f^\# \circ g^\#$ .
- (2)  $f_\# \dashv f^\#$  in **MRel**, that is,  $\Delta_X \subseteq f^\# \circ f_\#$  and  $f_\# \circ f^\# \subseteq \Delta_Y$ .

*Proof.* Regarding the first assertions, we calculate

$$g_{\#} \circ f_{\#} = g_{\#} \cdot Pf = e_Z^{\circ} \cdot \hat{P}g \cdot Pf = e_Z^{\circ} \cdot \hat{P}(g \cdot f) = (g \cdot f)_{\#}$$

and

$$f^{\#} \circ g^{\#} = f^{\circ} \cdot g^{\#} = f^{\circ} \cdot e_Y^{\circ} \cdot \hat{P}(g^{\circ}) = e_X^{\circ} \cdot \hat{P}(f^{\circ} \cdot g^{\circ}) = (g \cdot f)^{\#}.$$

To see (2), consider

$$f^{\#} \circ f_{\#} = f^{\#} \cdot Pf = e_X^{\circ} \cdot \hat{P}(f^{\circ}) \cdot Pf = e_X^{\circ} \cdot \hat{P}(f^{\circ} \cdot f) \supseteq e_X^{\circ} \cdot \hat{P}(1_X) = \Delta_X$$

and

$$f_{\#} \circ f^{\#} = \Delta_Y \cdot Pf \cdot \hat{P}(f^{\circ} \cdot \Delta_Y) \cdot m_Y^{\circ} = \Delta_Y \cdot Pf \cdot Pf^{\circ} \cdot \hat{P}(\Delta_Y) \cdot m_Y^{\circ} \subseteq \Delta_Y \cdot \hat{P}(\Delta_Y) \cdot m_Y^{\circ} = \Delta_Y \circ \Delta_Y = \Delta_Y.$$

□

Making use of Lemma 4 and of the adjunctions  $f \dashv f^{\circ}$  in  $\mathbf{Rel}$  and  $f_{\#} \dashv f^{\#}$  in  $\mathbf{MRel}$ , one obtains the following characterisations of consequence preserving maps.

**Proposition 6.** *Let  $f : X \rightarrow Y$  be a map between abstract logics  $(X, \vdash)$  and  $(Y, \Vdash)$ . Then*

$$\begin{aligned} f \text{ is consequence preserving} &\iff (f \cdot \vdash) \subseteq (\Vdash \cdot Pf) \\ &\iff \vdash \subseteq (f^{\circ} \cdot \Vdash \cdot Pf) \iff (\vdash \cdot Pf^{\circ}) \subseteq (f^{\circ} \cdot \Vdash) \\ &\iff \vdash \subseteq (f^{\#} \circ \Vdash \circ f_{\#}) \iff (\vdash \circ f^{\#}) \subseteq (f^{\#} \circ \Vdash) \\ &\iff (f_{\#} \circ \vdash) \subseteq (\Vdash \circ f_{\#}). \end{aligned}$$

Furthermore,  $f$  is conservative if and only if  $\vdash = (f^{\circ} \cdot \Vdash \cdot Pf)$  if and only if  $\vdash = (f^{\#} \circ \Vdash \circ f_{\#})$ .

A relation  $r : X \multimap Y$  is essentially the same thing as a function  $\lceil r \rceil : Y \rightarrow PX$ , via  $\lceil r \rceil(y) = \{x \in X \mid x r y\}$  and  $x r y \iff x \in \lceil r \rceil(y)$ . Therefore a relation  $r : PX \multimap Y$  corresponds to both a mapping

$$\mathcal{C}(r) : PX \rightarrow PY, A \mapsto \{y \in Y \mid A r y\}$$

and a mapping

$$\mathcal{U}(r) : Y \rightarrow PPX, y \mapsto \{A \subseteq X \mid A r y\}.$$

Furthermore,  $r : PX \multimap Y$  is monotone if and only if the map  $\mathcal{C}(r) : PX \rightarrow PY$  is monotone, if and only if the function  $\mathcal{U}(r) : Y \rightarrow PPX$  takes value in the set  $UX = \{\mathcal{A} \subseteq PX \mid \mathcal{A} \text{ is up-closed}\}$ . Here we call a subset  $\mathcal{A} \subseteq PX$  *up-closed* if  $A \in \mathcal{A}$  and  $A \subseteq B$  imply  $B \in \mathcal{A}$ . In the next two sections we will explore both point of views.

## 2. A TOPOLOGICAL VIEW ON LOGIC.

In the last subsection we have seen that every monotone relation  $r : PX \multimap Y$  corresponds precisely to a monotone mapping  $\mathcal{C}(r) : PX \rightarrow PY$ . Moreover, this transition preserves the compositional structure of monotone relations, as we show next.

**Proposition 7.** (1)  $\mathcal{C}(\Delta_X) = 1_{PX}$ , for every set  $X$ .

(2) Let  $r : X \multimap Y$  and  $s : Y \multimap Z$  be monotone relations. Then  $\mathcal{C}(s \circ r) = \mathcal{C}(s) \cdot \mathcal{C}(r)$ .

(3) Let  $r, r' : X \multimap Y$  be monotone relations. Then  $r \subseteq r'$  if and only if  $\mathcal{C}(r) \leq \mathcal{C}(r')$ .

- (4)  $\mathcal{C}(f_{\#}) = Pf$  and  $\mathcal{C}(f^{\#}) = Qf$ , for every map  $f : X \rightarrow Y$ . Here  $Qf : PY \rightarrow PX$ ,  $B \mapsto f^{-1}(B)$  is the right adjoint of  $Pf : PX \rightarrow PY$ .

*Proof.* It is useful here to think of a subset  $A \subseteq X$  as a monotone relation  $\alpha : \emptyset \multimap X$ , and with this interpretation  $\mathcal{C}(r) : PX \rightarrow PY$  is given by  $\alpha \mapsto r \circ \alpha$ . This proves at once the assertions (1)-(3). With the help of Lemma 4 it also tells us that  $\mathcal{C}(f^{\#}) = Qf$  and therefore, since  $f_{\#} \dashv f^{\#}$ , one gets  $\mathcal{C}(f_{\#}) \dashv Qf$ , hence  $\mathcal{C}(f_{\#}) = Pf$ .  $\square$

*Remark 8.* Formally, the proposition above states that  $\mathcal{C}$  is a functor whose domain is  $\mathbf{MRel}$  and whose codomain is the category with objects sets and morphisms monotone maps between the corresponding powersets. This is certainly related to the construction in [9, Section 4].

From this it follows at once that consequence relations  $\vdash$  on a set  $X$  correspond precisely to monotone maps  $c := \mathcal{C}(\vdash) : PX \rightarrow PX$  satisfying  $1_{PX} \leq c$  and  $c \cdot c \leq c$ , that is,

- (1)  $A \subseteq B \Rightarrow c(A) \subseteq c(B)$ ,
- (2)  $A \subseteq c(A)$ ,
- (3)  $c(c(A)) \subseteq c(A)$ ;

for all  $A, B \subseteq X$ . Note that one actually has equality in (3), thanks to (2). In generally, a function  $c : PX \rightarrow PX$  satisfying the conditions above is called a *closure operator*, and the pair  $(X, c)$  one calls a *closure space*.

A map  $f : X \rightarrow Y$  between closure spaces  $(X, c)$  and  $(Y, d)$  is called *continuous* whenever  $f$  preserves closure points in the sense that  $f(c(A)) \subseteq d(f(A))$ , for all  $A \subseteq X$ ; clearly,  $f$  is continuous if and only if it is consequence preserving with respect to the corresponding consequence relations. This can be equivalently expressed in the calculus of relations as  $Pf \cdot c \leq d \cdot Pf$  and, since  $Pf \dashv Qf$ , continuity of  $f$  is also equivalent to  $c \leq Qf \cdot d \cdot Pf$ . Furthermore, a continuous map  $f : X \rightarrow Y$  between closure spaces  $(X, c)$  and  $(Y, d)$  is called *initial* whenever  $c = Qf \cdot d \cdot Pf$ , which corresponds precisely to conservative maps of abstract logics. The connection with topology suggests yet another notion: we call a consequence preserving map  $f : X \rightarrow Y$  *open* whenever, for all  $x \in X$  and  $B \subseteq Y$  with  $B \vdash f(x)$ , there exists  $A \subseteq X$  with  $A \vdash x$  and  $f(A) \subseteq B$ . The designation “open” is motivated here by the formal similarity with the convergence description of open maps in topology (see [10], for instance). This condition translates into the inequality  $(f^{\#} \circ \vdash) \subseteq (\vdash \circ f^{\#})$ , hence, since  $(f^{\#} \circ \vdash) \supseteq (\vdash \circ f^{\#})$  follows from  $f$  being consequence preserving,  $f$  is open if and only if  $f^{\#} \circ \vdash = \vdash \circ f^{\#}$ .

### 3. A COALGEBRAIC VIEW ON LOGIC.

In this section we will think of an abstract logic  $\vdash$  on  $X$  as a mapping

$$\alpha := \mathcal{U}(\vdash) : X \rightarrow UX,$$

which brings us in the realm of coalgebras. This treatment of logic is motivated by fact that topological spaces can be seen as coalgebras for the filter functor [6], and the subsequent article [12] where closure systems are described as coalgebras for the “contravariant closure system functor”. However, our presentation differs slightly from [12] as we consider the up-set functor  $U$  (described below) which, moreover, is covariant. As we will see, the latter

is not an essential difference since  $Uf : UX \rightarrow UY$  has an adjoint  $Vf : UY \rightarrow UX$ , for every function  $f : X \rightarrow Y$ .

We recall that an  $F$ -coalgebra is a pair  $(A, \alpha)$  consisting of a set  $A$  and a function  $\alpha : A \rightarrow FA$ , where  $F$  is an endofunctor on  $\mathbf{Set}$  (or any other category). Given  $F$ -coalgebras  $(A, \alpha)$  and  $(B, \beta)$ , a *homomorphism* from  $(A, \alpha)$  to  $(B, \beta)$  is a function  $h : A \rightarrow B$  for which the square

$$\begin{array}{ccc} A & \xrightarrow{\alpha} & FA \\ h \downarrow & & \downarrow Fh \\ B & \xrightarrow{\beta} & FB \end{array}$$

commutes. The category of all  $F$ -coalgebras and homomorphisms is usual denoted by  $\mathbf{Coalg}(F)$ .

In general, the notion of coalgebra is in fact the dual of the notion of algebra, and therefore part of the mathematical theory of coalgebras can be obtained, by duality, from notions and properties of algebras, for instance the terminal-coalgebra construction can be obtained following the guidelines of the initial-algebra construction (see [1, Cor. 3.20]). However, since coalgebras have been successfully used as models of transition systems it is worth to study coalgebras by themselves. An important property, that will be used below, is that the forgetful functor  $\mathbf{Coalg}(F) \rightarrow \mathbf{Set}$  creates coproducts (see [1, Prop. 4.3]). That is, a coproduct of a coalgebras is obtained by equipping the corresponding coproduct of the carries of the coalgebras with the unique coalgebra structure turning the injections in homomorphisms.

In the sequel  $U$  denotes the up-set functor on  $\mathbf{Set}$ , where

$$UX = \{\mathcal{A} \subseteq PX \mid \mathcal{A} \text{ is up-closed}\},$$

and a function  $f : X \rightarrow Y$  is mapped to

$$Uf : UX \rightarrow UY, \mathcal{A} \mapsto \{B \subseteq Y \mid f^{-1}(B) \in \mathcal{A}\}.$$

We remark that the monotone map  $Uf : UX \rightarrow UY$  has a left adjoint  $Vf : UY \rightarrow UX$  defined by  $\mathcal{B} \mapsto \{f^{-1}(B) \mid B \in \mathcal{B}\}$ . It is useful to note that the functor  $U : \mathbf{Set} \rightarrow \mathbf{Set}$  comes together with the families of maps  $\eta_X : X \rightarrow UX$  and  $\mu_X : UUX \rightarrow UX$  ( $X$  is a set) defined by

$$\eta_X(x) = \{A \subseteq X \mid x \in A\} \quad \text{and} \quad \mu_X(\mathfrak{A}) = \{A \subseteq X \mid A^\# \in \mathfrak{A}\},$$

where  $A^\# = \{\mathcal{A} \in UX \mid A \in \mathcal{A}\}$ . In technical terms, the triple  $\mathbb{U} = (U, e, m)$  is a monad [7].

*Remark 9.* The construction  $f \mapsto f^\#$  defines actually a functor  $(-)^{\#} : \mathbf{Set} \rightarrow \mathbf{MRel}^{\text{op}}$  which is left adjoint to  $\text{hom}(-, 1) : \mathbf{MRel}^{\text{op}} \rightarrow \mathbf{Set}$ . The monad induced by this adjunction is precisely  $\mathbb{U}$ .

We also recall that the *Kleisli category*  $\mathbf{Set}_{\mathbb{U}}$  of  $\mathbb{U}$  has sets as objects, and a morphism  $\rho$  from  $Y$  to  $X$  in  $\mathbf{Set}_{\mathbb{U}}$  is a map  $\rho : Y \rightarrow UX$ . Given also  $\sigma : Z \rightarrow UY$ , their composite  $\rho * \sigma$  is defined as  $\rho * \sigma := \mu_X \cdot U\rho \cdot \sigma : Z \rightarrow UX$ ; and  $\eta_X : X \rightarrow UX$  is the identity morphism on  $X$  with respect to this composition. Moreover, we put  $\rho \leq \rho'$  whenever  $\rho(y) \subseteq \rho'(y)$

for all  $y \in Y$ , where  $\rho, \rho' : Y \rightarrow UX$ . Since the above-defined composition preserves this order from either side,  $\mathbf{Set}_{\mathbb{U}}$  is an ordered category as well. In fact:

**Proposition 10.**  $\mathcal{U} : \mathbf{MRel}^{\text{op}} \rightarrow \mathbf{Set}_{\mathbb{U}}$  is an equivalence of categories where, moreover,  $\mathcal{U}(r) \leq \mathcal{U}(r') \iff r \subseteq r'$ , for all monotone relations  $r, r' : PX \multimap Y$ .

*Proof.* One indeed verifies  $\mathcal{U}(s \circ r) = \mathcal{U}(r) * \mathcal{U}(s)$  for all monotone relations  $r : PX \multimap Y$  and  $PY \multimap Z$ , as well as  $\mathcal{U}(\Delta_X) = \eta_X$ .  $\square$

**Corollary 11.** A map  $\alpha : X \rightarrow UX$  comes from a consequence relation  $\vdash$  on  $X$  if and only if

$$\eta_X(x) \subseteq \alpha(x) \quad \text{and} \quad \mu_X \cdot U\alpha \cdot \alpha(x) \subseteq \alpha(x),$$

for all  $x \in X$ , that is,  $\eta_X \leq \alpha$  and  $\alpha * \alpha \leq \alpha$ .

As before, the second inequality above is necessarily an equality thanks to the first inequality.

For  $f : X \rightarrow Y$ , we define maps

$$\begin{aligned} f_{\diamond} : X &\rightarrow UY & \text{and} & & f^{\diamond} : Y &\rightarrow UX \\ x &\mapsto \{B \subseteq Y \mid f(x) \in B\} & & & y &\mapsto \{A \subseteq X \mid y \in f(A)\}, \end{aligned}$$

and one has

$$\mathcal{U}(f_{\#}) = f^{\diamond} \quad \text{and} \quad \mathcal{U}(f^{\#}) = f_{\diamond}.$$

From Lemma 4 we obtain that, with  $\rho : Y \rightarrow UZ$  and  $\sigma : Z \rightarrow UY$ ,

$$\rho * f_{\diamond} = \rho \cdot f \quad \text{and} \quad f^{\diamond} * \sigma = Vf \cdot \sigma,$$

and therefore (c.f. Corollary 5)  $(g \cdot f)_{\diamond} = g_{\diamond} * f_{\diamond}$  and  $(g \cdot f)^{\diamond} = f^{\diamond} * g^{\diamond}$ , where  $g : Y \rightarrow Z$ , and  $f_{\diamond} \dashv f^{\diamond}$  in  $\mathbf{Set}_{\mathbb{U}}$ .

Let now  $(X, \vdash)$  and  $(Y, \Vdash)$  abstract logics with corresponding maps  $\alpha : X \rightarrow UX$  and  $\beta : Y \rightarrow UY$ . Then  $f$  is consequence preserving if and only if

$$\alpha * f^{\diamond} \leq f^{\diamond} * \beta,$$

which is equivalent to

$$f_{\diamond} * \alpha \leq \beta * f_{\diamond},$$

and this in turn reduces to  $Uf \cdot \alpha \leq \beta \cdot f$ . Moreover,  $f$  is conservative if and only if  $\alpha = f^{\diamond} * \beta * f_{\diamond}$ , or, equivalently  $\alpha = Vf \cdot \beta \cdot f$ . Somehow dually, we say that  $f$  is *progressive* if  $\beta = f_{\diamond} * \alpha * f^{\diamond}$ . Note that conservative maps are the coalgebra morphisms in the sense of [12], moreover, every conservative map as well as every progressive map is consequence preserving. Finally,  $f$  is open if and only if  $Uf \cdot \alpha = \beta \cdot f$ , that is,  $f$  is a morphism of coalgebras. Also note that every open injection is conservative and every open surjection is progressive.

To finish this section, we apply the internal characterisation above and show that the class of coalgebras induced by an abstract logic is a *covariety*, that is, it is closed under homomorphic images, subcoalgebras and sums.

**Lemma 12.** Let  $(X, \alpha)$  and  $(Y, \beta)$  be coalgebras and  $f : X \rightarrow Y$  be a map. Then



- (1)  $\alpha$  is induced by an abstract logic provided that  $f$  is conservative and  $\beta$  is induced by an abstract logic.
- (2)  $\beta$  is induced by an abstract logic if  $f$  is progressive and  $\alpha$  is induced by an abstract logic.

*Proof.* To see (1), just note that

$$\alpha = f^\diamond * \beta * f_\diamond \supseteq f^\diamond * \eta_Y * f_\diamond = f^\diamond * f_\diamond \supseteq \eta_X$$

and

$$\alpha * \alpha = f^\diamond * \beta * f_\diamond * f^\diamond * \beta * f_\diamond \subseteq f^\diamond * \beta * \beta * f_\diamond \subseteq f^\diamond * \beta * f_\diamond = \alpha.$$

The second statement can be proven in a similar way.  $\square$

**Theorem 13.** *The class of coalgebras induced by an abstract logic is a covariety.*

*Proof.* The previous lemma implies at once that the class of coalgebras induced by an abstract logic is closed under homomorphic images and subcoalgebras. To show closedness under the formation of sums, we note that the sum of a family  $(X_i, \vdash_i)_{i \in I}$  of abstract logics can be calculated as the disjoint union  $X = \coprod_{i \in I} X_i$ , equipped with the consequence relation  $\vdash$  defined by  $A \vdash x$  whenever  $(A \cap X_i) \vdash_i x$ , where  $x \in X_i$  (see also [8]). By definition, every inclusion map  $k_i : (X_i, \vdash_i) \hookrightarrow (X, \vdash)$  is open. Hence,  $\alpha := \mathcal{U}(\vdash)$  is a coalgebra structure on  $X$  making every  $k_i : (X_i, \alpha_i) \hookrightarrow (X, \alpha)$  (where  $\alpha_i := \mathcal{U}(\vdash_i)$ ) a coalgebra morphism, and this tells us that the coalgebra  $(X, \alpha)$  is the sum of  $(X_i, \alpha_i)_{i \in I}$ .  $\square$

*Remark 14.* The methods employed in this paper can be applied also in other contexts. For instance, if one starts in Section 1 with the ultrafilter monad instead of the powerset monad, then an abstract logic “becomes” a topological space (see [2]), and the adjunction of Remark 9 induces the filter monad. In this technical sense, our presentation of abstract logics as coalgebras is the same as the one for topological spaces. In a similar way, we might use the identity monad, then Section 1 talks about preordered sets (i.e. sets equipped with a reflexive and transitive but not necessarily anti-symmetric relation) and the adjunction of Remark 9 induces the powerset monad.

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